

On Hunting for Taxicab Numbers

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February 11, 2008

Abstract

In this article, we make use of some known method to investigate some properties of the numbers represented as sums of two equal odd powers, i.e., the equation $x^n + y^n = N$ for $n \geq 3$. It was originated in developing algorithms to search new taxicab numbers (i.e., naturals that can be represented as a sum of positive cubes in many different ways) and to verify their minimality. We discuss properties of diophantine equations that can be used for our investigations. This techniques is applied to develop an algorithm allowing us to compute new taxicab numbers (i.e., numbers represented as sums of two positive cubes in k different ways), for $k = 7 \dots 14$.

Introduction

This work was originated in searching new so-called *taxicab numbers*, i.e., naturals T_k that can be represented/decomposed as/into a sum of positive cubes in k different ways, and verifying their minimality. We made use of some known method to investigate properties of the cubic equation that could help us to find new taxicab numbers.

Already Fermat proved that numbers expressible as a sum of two cubes in n different ways exist for any n . But still finding taxicab numbers and proving their minimality are hard computational problems. Whereas the first nontrivial taxicab number $T_2 = 1729$ became widely-known in 1917 thanks to Ramanujan and Hardy, next ones were only found with help of computers: $T_3 = 87539319$ (J. Leech, 1957), $T_4 = 6963472309248$ (E. Rosenstiel, J.A. Dardis, and C.R. Rosenstiel, 1991), $\mathbf{W}_5 = T_5 = 48988659276962496$ (D. Wilson, 1997, [7]). It is known that these numbers are minimal. For $\mathbf{R}_6 = T_6 = 24153319581254312065344$ (R.L. Rathbun, 2002) as well as for next discovered taxicab numbers it is unknown.

In January–September 2006 the author computed $T_7 = 139^3 \mathbf{R}_6$, $T_8 = 727^3 T_7$, $T_9 = 4327^3 T_8$, $T_{10} = 38623^3 T_9$, and $T_{11} = 45294^3 T_{10}$. At the end

of 2006 the author learned about the results of C. Boyer [1] who established smaller T_7, \dots, T_{11} and first T_{12} in December 2006. At the begin of 2007 the author computed T_{13} and T_{14} .

The article is organized as follows. We start with putting the equation in a new form. Next, we deduce simple properties of the equation of interest based on this presentation. At the end, we present a new algorithm to compute taxicab numbers which we used to find new ones.

1 Common Properties

We are interested in the problem of representations (also called decompositions) of numbers as the sums of two positive odd n -powers; i.e., solvability of the equation

$$x^n + y^n = N \quad (1)$$

in positive integers. A solution of this equation is also called a representation or a decomposition of the number N . The equation of interest is too “smooth” in its original form. We want to make it “uneven”. We are going to consider this equation in the following $m \pm h$ -form ($m \neq h > 0$)

$$(m - h)^n + (m + h)^n = N \quad (2)$$

which is not an infrequent guest in number-theoretical proofs.

Although only even numbers can be directly represented in this way, there is a simple transformation that allows us to treat this equation for odd N as well. In fact, any pair (x, y) consisting of even and odd integers can be represented as $(t - s - 1, t + s)$. If N is odd, we write

$$(t - s - 1)^n + (t + s)^n = N.$$

Multiplying both sides by 2^n we can put the previous equation into the form

$$((2t - 1) - (2s + 1))^n + ((2t - 1) + (2s + 1))^n = 2^n N \quad (3)$$

and, then some extra steps are needed to obtain representations of N itself. For the exponent 3, the least odd number N for which $2^3 N$ yields a not only proper two cubes representation is 513:

$$2^3 513 = 2^3 (1^3 + 8^3) = (12 - 3)^3 + (12 + 3)^3 = 4104.$$

Notice that 4104 is the least even number represented as a sum of two cubes in two different ways. Next, assume N to be even if we do not explicitly state the contrary.

We are interested in any prime powers, although sometimes it is sufficient that they are odd only. Such representations for odd powers are closely related to divisors of the numbers of interest.

We shall refer to m as a *median* of the corresponding power representation and to N_d as an integer quotient N/d if it exists. We shall make use the following property (a simple corollary of Quadratic Reciprocity Law) of odd prime divisors of binary forms:

Property 1.1.

$$p \mid ax^2 + by^2 \wedge \gcd(ax, by) = 1 \implies \left(\frac{ab}{p} \right) = (-1)^{\frac{p-1}{2}}.$$

In particular, for the binary form $u^2 + 3v^2$ the forbidden divisors are

$$\left(\frac{3}{p} \right) \neq (-1)^{\frac{p-1}{2}}; \quad \text{i.e., } p \equiv 5, 11 \pmod{12}.$$

Given N and its divisors, by solving an $n - 1$ -order polynomial equation

$$(m - h)^n + (m + h)^n = 2m \left(\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} m^{n-2k-1} h^{2k} \right) = N \quad (4)$$

with respect to h , we can either “easily” find some representation(s) of this number or prove that it is impossible. Notice that in this polynomial m and h occur only in odd and even powers, respectively.

We start the investigation by establishing the following simple properties of **Equation (2)**.

Lemma 1. *If m is a median of some representation of N , then*

$$m \equiv N_2 \pmod{n}.$$

If $n \mid N$, then also $n^2 \mid N$. If $n \nmid N$, then $N = 2m(nt + 1)$.

Proof. First, rewriting **Equation (2)** in the form

$$m^n + n \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n} \binom{n}{2k} m^{n-2k} h^{2k} = N_2$$

we can derive the modular equation $m^n \equiv N_2 \pmod{n}$. Next:

- By applying Fermat’s Little Theorem we have the first statement.
- Because $n \mid N$, therefore also $n \mid m$, and this yields the second statement.
- Because $n \nmid N$, then also $n \nmid m$. By applying Fermat’s Little Theorem to $m^{n-1} \equiv N_{2m} \pmod{n}$ we have the third statement.

□

Because h is ranged in $(0, m)$ it is easy to establish

Lemma 2. *A necessary condition for N to have a representation as the sum of n -powers is*

$$\exists m \mid N : \sqrt[n]{\frac{N}{2^n}} < m < \sqrt[n]{\frac{N}{2}}.$$

Obviously, the number of such representations does not exceed the number of divisors of N satisfying this condition (see also **Lemma 3**).

Lemmas 1 and **2** allow us to estimate numbers being the sum of two odd powers higher than 2 in k ways. If a number has two different representations for the power n , then the medians m_1, m_2 corresponding to them also satisfy the congruence $m_1 \equiv m_2 \pmod{n}$. Because $\sqrt[n]{N/2^n} + n(k-1) \leq \sqrt[n]{N/2}$, we have the following properties of generalized taxicab numbers

Lemma 3. *If number $T(n, k)$, $k > 1$, represented as the sum of two n -powers in k ways is even, then it has at least k divisors in the range $(\sqrt[n]{N/2^n}, \sqrt[n]{N/2})$ and the following lower bound holds*

$$T(n, k) \geq 2 \left(\frac{2n}{2 - \sqrt[n]{2}} \right)^n (k-1)^n$$

This bound is far from optimal due to a quite conservative assumption about the gaps between medians. This is a subject of further investigation. Recall that only wide-known theoretical bound for $T(3, k) = T_k$ is Silverman's result [6] that describes its logarithmic behavior:

$$\log T_k = o(k^{r+2/r}),$$

where r is the highest rank of **Equation (1)**. The highest rank known now is 5.

When there are "too many" taxicab medians they cannot be relative prime because all of them are divisors. Hence they share common divisors. In particular, for taxicab medians $m_1 < \dots < m_k$ the following inequality holds:

$$\text{lcm}(m_1, \dots, m_k) \leq (2m_1)^n.$$

The cubic equation in the form $m^2 + 3h^2 = N_{2m}$ provides a way to derive parameterizations of the two cubes representation problem¹. We mention only those of them that relate to the taxicab numbers problem. It arises when N_{2m} is a cube and this case is connected to the well-known problem of the decomposition of numbers into two rational cubes (positive or not) which was investigated by Fermat, Euler, Sylvester, and other researchers.

Gérardin proved [4, Chapter XX] that all solutions of $u^2 + 3v^2 = w^3$ with $\gcd(u, v) = 1$ are generated by

$$(t^3 - 9ts^2)^2 + 3(3t^2s - 3s^3)^2 = (t^2 + 3s^2)^3.$$

We have

$$(t^3 - 3t^2s - 9ts^2 + 3s^3)^3 + (t^3 + 3t^2s - 9ts^2 - 3s^3)^3 = 2(t^3 - 9ts^2)(t^2 + 3s^2)^3,$$

and next

$$\left(\frac{t^3 - 3t^2s - 9ts^2 + 3s^3}{t^2 + 3s^2} \right)^3 + \left(\frac{t^3 + 3t^2s - 9ts^2 - 3s^3}{t^2 + 3s^2} \right)^3 = 2t(t-3s)(t+3s).$$

¹Here we treat independently the median m and its co-factor N_{2m} ; therefore this does not cover general cases.

So, if the diophantine equation $2t^3 - 18ts^2 \pm Nr^3 = 0$ is solvable², then N is decomposable.

This can be simplified into one-parametric examples as follows

$$\left(\frac{w^3 + 3w^2 - 6w + 1}{3(w^2 - w + 1)} \right)^3 - \left(\frac{w^3 - 6w^2 + 3w + 1}{3(w^2 - w + 1)} \right)^3 = w(w - 1),$$

and

$$\pm \left(\frac{8w^9 \pm 24w^6 + 6w^3 \mp 1}{3w(4w^6 \pm 2w^3 + 1)} \right)^3 \mp \left(\frac{8w^9 \mp 12w^6 - 12w^3 \mp 1}{3w(4w^6 \pm 2w^3 + 1)} \right)^3 = 4w^3 \pm 2.$$

Also, the substitution $t - 3s = u^2v, t + 3s = uv^2$ gives

$$\left(\frac{u^3 + 6u^2v + 3uv^2 - v^3}{3(u^2 + uv + v^2)} \right)^3 + \left(\frac{v^3 + 6v^2u + 3vu^2 - u^3}{3(u^2 + uv + v^2)} \right)^3 = uv(u + v)$$

which provides the following parametrization of the sum of two integer powers

$$\left(\frac{p^9 + 6p^6q^3 + 3p^3q^6 - q^9}{3pq(p^6 + p^3q^3 + q^6)} \right)^3 + \left(\frac{q^9 + 6q^6p^3 + 3q^3p^6 - p^9}{3pq(p^6 + p^3q^3 + q^6)} \right)^3 = p^3 + q^3.$$

Catalan's parametrization

$$\left(\frac{1}{2}(t + s)(t - 2s)(s - 2t) \right)^2 + 3 \left(\frac{3}{2}ts(t - s) \right)^2 = (t^2 - ts + s^2)^3$$

leads us to another rational cubes identity

$$\left(\frac{t^3 - 3t^2s + s^3}{t^2 - ts + s^2} \right)^3 + \left(\frac{t^3 - 3ts^2 + s^3}{t^2 - ts + s^2} \right)^3 = (t + s)(2s - t)(s - 2t).$$

The substitution $2s - t = u^2v, s - 2t = uv^2$ gives the following identity

$$\left(\frac{u^3 + 3u^2v - 6uv^2 + v^3}{3(u^2 - uv + v^2)} \right)^3 - \left(\frac{u^3 - 6u^2v + 3uv^2 + v^3}{3(u^2 - uv + v^2)} \right)^3 = uv(u - v)$$

which provides the following parametrization of the sum of two integer powers

$$\left(\frac{p^9 + 3p^6q^3 - 6p^3q^6 + q^9}{3pq(p^6 - p^3q^3 + q^6)} \right)^3 - \left(\frac{p^9 - 6p^6q^3 + 3p^3q^6 + q^9}{3pq(p^6 - p^3q^3 + q^6)} \right)^3 = p^3 - q^3.$$

It is easy to note that these parameterizations of the sum and the difference of two integer cubes also give parameterizations to the diophantine equation $X^3 + Y^3 = S^3 + T^3$. Euler's parametric solution to $X^3 + Y^3 = S^3 + T^3$ is

$$\begin{aligned} X &= w(1 - (u - 3v)(u^2 + 3v^2)) & Y &= w((u + 3v)(u^2 + 3v^2) - 1) \\ S &= w((u + 3v) - (u^2 + 3v^2)^2) & T &= w((u^2 + 3v^2)^2 + (3v - u)) \end{aligned}$$

Finally, we mention some properties of the equation of interest that can be used to investigate taxicab numbers. Sometimes we can improve the congruence of **Lemma 1**:

²Euler's solution of the two rational cubes problem is slightly different.

Lemma 4. *If $(m-h)^p + (m+h)^p = N$, $\gcd(m, h) = 1$, $m \not\equiv h \pmod{2}$, then*

$$p = 3 \Rightarrow m \equiv N_2 \pmod{12}$$

$$p = 5 \Rightarrow m \equiv N_2 \pmod{20}.$$

Proof. We write down $2m^3 + 6mh^2 = N$ as $m^2 - h^2 + 4h^2 = N_{2m}$ and $2m^5 + 20m^3h^2 + 10mh^4 = N$ as $5m(m^2 + h^2)^2 - 4m^5 = N_2$. Considering these equations by modulo 4 we conclude $m \equiv N_2 \pmod{4}$. Combining this congruence with the congruence from **Lemma 1** we obtain these lemma statements. \square

The forbidden divisors condition for two-squares representation is well known since Fermat's work. For cubic and quintic equations there are analogies which follow from **Property 1.1**:

Lemma 5. *Necessary conditions for N to have a cubic/quintic representation with $\gcd(m, h) = 1$ are the following:*

1. *It has no prime divisors of forms $12t + 5$ and $12t + 11$ (the cubic case) or of forms $10t \pm 1$ (the quintic case), or*
2. *If such divisors exist, then all of them are factors of the median.*

Remark. In view of the cubic case of **Lemma 5**, we can mention the results of Euler et al for the divisors of numbers in the form $u^2 + 3v^2$: all prime divisors have the same form $\alpha^2 + 3\beta^2$.

2 New Taxicab/Cabtaxi Numbers

Before we discuss cubic taxicab numbers, we briefly consider the equation $x^5 + y^5 = u^5 + v^5$. No such number is known within the range up to $1.05 \cdot 10^{26}$. We have not yet found any, but we found some solution in Gaussian integers:

$$\begin{aligned} & (t^2 + s^2 - (t^2 - 2ts - s^2)\iota)^5 + (t^2 + s^2 + (t^2 - 2ts - s^2)\iota)^5 = \\ & (t^2 + s^2 - (t^2 + 2ts - s^2)\iota)^5 + (t^2 + s^2 + (t^2 + 2ts - s^2)\iota)^5 = \\ & -8(t^2 + s^2)(t^4 - 2t^3s - 6t^2s^2 + 2ts^3 + s^4)(t^4 + 2t^3s - 6t^2s^2 - 2ts^3 + s^4) \end{aligned}$$

The least such positive number is $3800 = (5 - \iota)^5 + (5 + \iota)^5 = (5 - 7\iota)^5 + (5 + 7\iota)^5$.

The observation that $T_6 = 79^3 T_5$ stirs up our interest in searching for new taxicab numbers T_k in the same way. The usual definition of taxicab numbers is equipped with a condition that they are minimal. But for brevity we designate all multi-ways representable numbers as taxicab numbers. Even an open question³ about the minimality of T_6 does not matter. To compute some $k+1$ -way representable number we can try any k -way representable number. Our

³C. Calude et al [2] (with an update [3]) stated that the minimality of T_6 can be confirmed with the probability > 0.99 but G. Martin criticized their considerations in Mathematical Reviews MR2149410 (2006a:11175).

approach can produce non-minimal numbers, but such numbers can be used to check their minimality or to search for smaller ones. We believe that this median-based approach reducing the length of tested numbers in three times allows us to check the minimality of T_6 and T_7 .

Notice that Wilson [7] used similar ideas (cubic multipliers) to find 5-way representable number $\mathbf{W}_5 = 48988659276962496$ in 1997 but his approach is more expensive even for small numbers. During this search a six-way example was also detected. Inspired by Wilson's approach in 2002 R. L. Rathbun [5] presented the smaller candidate

$$\mathbf{R}_6 = 79^3 \mathbf{W}_5 = 24153319581254312065344.$$

Rathbun also mentioned multipliers 139 and 727 giving other examples of six-way representable numbers. Our approach demonstrates that they appear in multipliers of T_9 and T_{11} , respectively.

In the first version of this article (December 2006) we described a modification of our algorithm that produces some taxicab numbers. In January–September 2006 with help of this algorithm we computed $T_7 = 139^3 \mathbf{R}_6$, $T_8 = 727^3 T_7$, $T_9 = 4327^3 T_8$, $T_{10} = 38623^3 T_9$, and $T_{11} = 45294^3 T_{10}$. At that moment we learned about results of C. Boyer [1] who established smaller T_7, \dots, T_{11} and first T_{12} in December 2006. Unfortunately he has not yet published details of his algorithm. Our renewed algorithm, given later in this article, produces the same numbers. Also, for the first time we found T_{13} and T_{14} .

The main idea of our approach is not too surprising. If we know some k -way representable number T_k , then we can try to find T_{k+1} in the form $\mu^3 T_k$. If m_1, \dots, m_k are medians of the representations of T_k , then medians of the representations of T_{k+1} are $\mu m_1, \dots, \mu m_k, d'd$ where $d' \in \text{divisors}(\mu^3)$ (the first version of the algorithm uses only $d' = 1$) and $d \in \text{divisors}(T_k)$. A simple observation is that the multiplier of interest does not exceed $2T_k^{2/3}$.

The iterative procedure formalizing this idea and using the properties of the equation is the following:

- Create an ordered array D of all divisors of T_k excluding known too small divisors, i.e., less than $\sqrt[3]{T_k/4}$.
- For multipliers M from 2 to $\lfloor 2T_k^{2/3} \rfloor$ do
 - Let $N = M^3 T_k$;
 - For $\mu \in \text{divisors}(M^3)$ do
 - Using dichotomic search, find a range of D where the divisors satisfying **Lemma 2** for $\frac{1}{\mu}N$ are located;
 - Within this range for divisors d such that $\mu d \equiv \frac{1}{2}N \pmod{3}$ do: if the value $(\frac{1}{2\mu d}N - (\mu d)^2)/3$ is a perfect square, then μd is the $k+1$ median and therefore N is T_{k+1} . Otherwise continue.

A set of all divisors of T_k may be space-consuming. To avoid the explicit computation of this set we used the following trick. A taxicab number T_k is a product $M \cdot T_s$ where T_s is a “seed”, i.e., a small taxicab number with an easily computed set of divisors and $M = (\mu_{s+1} \cdots \mu_k)^3$. Evidently $M = 1$ for $T_{k+1} = T_{s+1}$. Thus computing T_{k+1} we split the loop iterating through all divisors of T_k into two nested loops: the outer loop iterating through all divisors of M and the inner one iterating through those divisors of T_s such that product of the first iterator, the second iterator, and some divisor of the current cubic multiplier satisfies **Lemma 2**.

Choice of the seed T_s affects the space used by the algorithm. We used W_5 to compute new T_k for $k = 7 \dots 12$. But for the next numbers, cardinality of the divisor set for M exceeds one for T_s more and more. To balance the cardinalities of these sets we took greater seeds.

Ways	Seed	Multiplier	Time
7	5	101	58 s.
8	5	127	5 m. 1 s.
9	5	139	18 m. 47 s.
10	5	377	4 h. 8 m.
11	5	727	123 h. 20 m.
12	5	2971	152 d.
13	6	4327	21 h. 8 m. ^{a)}
14	6	7549	23 m. 39 s. ^{b)}

^{a)} To compute this number we examined only prime multipliers great than 2971.

^{b)} To compute this number we examined only this multiplier.

Table 1. Computational results.

Table 1. represents multipliers producing new taxicab numbers. In **APPENDIX A** we give these numbers themselves and their decompositions.

Also, we found that all of our taxicab numbers $T(3, k)$ are *cabtaxi* (i.e., without the restriction on the cubes of the decomposition to be positive) numbers $C(3, k+2)$. Surprisingly the multiplier 5 gives cabtaxi numbers of higher orders: $5^3 T(3, k) = C(3, k+4)$. We checked this property for $k = 6 \dots 12$.

Final Remark

In September 2007 we learned about new results of C. Boyer who established new taxicab numbers for $n = 13 \dots 19$ and cabtaxi numbers for $n = 10 \dots 30$. Boyer’s article is going to be published in a mathematical magazine.

References

- [1] C. Boyer, *New upper bounds of taxicab and cabtaxi numbers*, December 8, 2006, <http://cboyer.club.fr/Taxicab.htm>.
- [2] C. S. Calude, E. Calude, and M. J. Dinneen, *What is the value of Taxicab(6)?*, Journal of Universal Computer Science **9** (2003), 1196–1203.
- [3] ———, *What is the value of Taxicab(6)? An update*, Research Report CDMTCS-261, University of Auckland and Massey University at Albany, NZ, 2005.
- [4] L. E. Dickson, *History of the Theory of Numbers. Volume II: Diophantine Analysis*, AMS Chelsea Publishing Company, Providence, Rhode Island, 1999.
- [5] R. L. Rathbun, *Sixth taxicab number?*, July 16, 2002, <http://listserv.nodak.edu/scripts/wa.exe?A2=ind0207&L=nmbrrthry&P=R530>.
- [6] J. H. Silverman, *Integer points on curves of genus 1*, Journal of London Mathematical Society **28** (1983), 1–7.
- [7] D. W. Wilson, *The fifth Taxicab number is 48988659276962496*, Journal of Integer Sequences **2** (1999), Article 99.1.9.

APPENDIX A. Taxicab numbers decompositions

$T_7 = 101^3 R_6 = 24885189317885898975235988544$:

$$\begin{array}{rcl}
 58798362^3 & + & 2919526806^3 = \\
 309481473^3 & + & 2918375103^3 = \\
 459531128^3 & + & 2915734948^3 = \\
 860447381^3 & + & 2894406187^3 = \\
 1638024868^3 & + & 2736414008^3 = \\
 1766742096^3 & + & 2685635652^3 = \\
 1847282122^3 & + & 2648660966^3
 \end{array}$$

$T_8 = 127^3 T_7 = 50974398750539071400590819921724352$:

$$\begin{array}{rcl}
 7467391974^3 & + & 370779904362^3 = \\
 39304147071^3 & + & 370633638081^3 = \\
 58360453256^3 & + & 370298338396^3 = \\
 109276817387^3 & + & 367589585749^3 = \\
 208029158236^3 & + & 347524579016^3 = \\
 224376246192^3 & + & 341075727804^3 = \\
 234604829494^3 & + & 336379942682^3 = \\
 288873662876^3 & + & 299512063576^3
 \end{array}$$

$$T_9 = 139^3 T_8 = 136897813798023990395783317207361432493888:$$

$$\begin{array}{rcl} 1037967484386^3 & + & 51538406706318^3 = \\ 4076877805588^3 & + & 51530042142656^3 = \\ 5463276442869^3 & + & 51518075693259^3 = \\ 8112103002584^3 & + & 51471469037044^3 = \\ 15189477616793^3 & + & 51094952419111^3 = \\ 28916052994804^3 & + & 48305916483224^3 = \\ 31188298220688^3 & + & 47409526164756^3 = \\ 32610071299666^3 & + & 46756812032798^3 = \\ 40153439139764^3 & + & 41632176837064^3 \end{array}$$

$$T_{10} = 377^3 T_9 = 7335345315241855602572782233444632535674275447104:$$

$$\begin{array}{rcl} 391313741613522^3 & + & 19429979328281886^3 = \\ 904069333568884^3 & + & 19429379778270560^3 = \\ 1536982932706676^3 & + & 19426825887781312^3 = \\ 2059655218961613^3 & + & 19422314536358643^3 = \\ 3058262831974168^3 & + & 19404743826965588^3 = \\ 5726433061530961^3 & + & 19262797062004847^3 = \\ 10901351979041108^3 & + & 18211330514175448^3 = \\ 11757988429199376^3 & + & 17873391364113012^3 = \\ 12293996879974082^3 & + & 17627318136364846^3 = \\ 15137846555691028^3 & + & 15695330667573128^3 \end{array}$$

$$T_{11} = 727^3 T_{10} = 2818537360434849382734382145310807703728251895897826621632:$$

$$\begin{array}{rcl} 284485090153030494^3 & + & 14125594971660931122^3 = \\ 657258405504578668^3 & + & 14125159098802697120^3 = \\ 1117386592077753452^3 & + & 14123302420417013824^3 = \\ 1497369344185092651^3 & + & 14120022667932733461^3 = \\ 2223357078845220136^3 & + & 14107248762203982476^3 = \\ 4163116835733008647^3 & + & 14004053464077523769^3 = \\ 6716379921779399326^3 & + & 13600192974314732786^3 = \\ 7925282888762885516^3 & + & 13239637283805550696^3 = \\ 8548057588027946352^3 & + & 12993955521710159724^3 = \\ 8937735731741157614^3 & + & 12815060285137243042^3 = \\ 11005214445987377356^3 & + & 11410505395325664056^3 \end{array}$$

$$T_{12} = 2971^3 T_{11} =$$

73914858746493893996583617733225161086864012865017882136931801625152:

$$\begin{aligned}
& 845205202844653597674^3 + 41967142660804626363462^3 = \\
& 1933097542618122241026^3 + 41965889731136229476526^3 = \\
& 1952714722754103222628^3 + 41965847682542813143520^3 = \\
& 3319755565063005505892^3 + 41960331491058948071104^3 = \\
& 4448684321573910266121^3 + 41950587346428151112631^3 = \\
& 6605593881249149024056^3 + 41912636072508031936196^3 = \\
& 12368620118962768690237^3 + 41606042841774323117699^3 = \\
& 19954364747606595397546^3 + 40406173326689071107206^3 = \\
& 23546015462514532868036^3 + 39334962370186291117816^3 = \\
& 25396279094031028611792^3 + 38605041855000884540004^3 = \\
& 26554012859002979271194^3 + 38073544107142749077782^3 = \\
& 32696492119028498124676^3 + 33900611529512547910376^3
\end{aligned}$$

$T_{13} = 4327^3 T_{12} =$

5988146776742829080553965820313279739849705084894534523771076163371248442670016:

$$\begin{aligned}
& 3657202912708816117135398^3 + 181591826293301618274700074^3 = \\
& 8364513066908614936919502^3 + 181586404866626464944928002^3 = \\
& 8449396605357004644311356^3 + 181586222922362752472011040^3 = \\
& 14364582330027624823994684^3 + 181562354361812068303667008^3 = \\
& 19249457059450309721505567^3 + 181520191447994609864354337^3 = \\
& 28582404724165067827090312^3 + 181355976285742254187920092^3 = \\
& 53519019254751900122655499^3 + 180029347376357496130283573^3 = \\
& 54818831102057750995052604^3 + 179911586979069103444414128^3 = \\
& 86342536262893738285181542^3 + 174837511984583610680880362^3 = \\
& 101883608906300383719991772^3 + 170202382175796081666789832^3 = \\
& 109889699639872260803223984^3 + 167044016106588827404597308^3 = \\
& 114899213640905891306456438^3 + 164744225351606675259562714^3 = \\
& 141477721399036311385473052^3 + 146687946088200794808196952^3
\end{aligned}$$

$T_{14} = 7549^3 T_{13} =$

$$\begin{aligned}
& 257608810925730001281963766003343299028977072 \setminus \\
& 5881505682307757452553496715044742867424072384 :
\end{aligned}$$

27608224788038852868255119502 ³	+	1370836696688133916355710858626 ³	=
63143709142093134158805320598 ³	+	1370795770338163183869261487098 ³	=
63784494973840028059906426444 ³	+	1370794396840916418411211340960 ³	=
108438232009378539796335869516 ³	+	1370614213077319303624382243392 ³	=
145314151341790388087645525283 ³	+	1370295925240911309866010890013 ³	=
215768573262722097026704765288 ³	+	1369056264981068276864608774508 ³	=
404015076354122094025926361951 ³	+	1359041543344122738287510692577 ³	=
413827355989433962261652107596 ³	+	1358152570104992661901882252272 ³	=
617989830682279948575932296880 ³	+	1327627770274178602420131034444 ³	=
651799806248584830314835460558 ³	+	1319848377971621677029965852738 ³	=
769119363633661596702217886828 ³	+	1284857783045084620502596441768 ³	=
829557342581395696803537855216 ³	+	1261015277588639058077305078092 ³	=
867374163775198573472439650462 ³	+	1243654157179278791534438927986 ³	=
1068015318841325114648936069548 ³	+	1107347305019827800007078790648 ³	=